B.Sc. Part III (H) 7th Paper D. Eqns

Q: Solve $1+p^2 = qz$.

Soln. The given equation

$$1+p^2 = qz. \quad \text{--- (1)}$$

It is of the form $f(p, q, z) = 0$.

Put $u = x + ay$

$$\Rightarrow \frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = a.$$

$$\therefore p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du}.$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} = a\frac{dz}{du}$$

Putting these values in (1), we get

$$1 + \left(\frac{dz}{du}\right)^2 = az\frac{dz}{du}$$

$$\Rightarrow \left(\frac{dz}{du}\right)^2 - az\frac{dz}{du} + 1 = 0$$

$$\Rightarrow \frac{dz}{du} = \frac{az \pm \sqrt{a^2z^2 - 4}}{2}$$

$$\Rightarrow \frac{dz}{az \pm \sqrt{a^2z^2-4}} = \frac{du}{2}$$

$$\Rightarrow \frac{az \mp \sqrt{a^2z^2-4}}{(az \pm \sqrt{a^2z^2-4})(az \mp \sqrt{a^2z^2-4})}dz = \frac{du}{2}$$

$$\Rightarrow \frac{az \mp \sqrt{a^2z^2-4}}{a^2z^2-(a^2z^2-4)} \times 2dz = du$$

$$\Rightarrow \left(az \mp \sqrt{a^2z^2-4}\right) dz = 2du$$

Integrating, we get

$$\Rightarrow a\int z\,dz \mp \int \sqrt{a^2z^2-4}\,dz = 2\int du$$

Integrating, we get

$$\Rightarrow \cancel{a\frac{z^2}{2} \mp \left[\frac{az}{2}\sqrt{a^2z^2-4} - 4\log\left\{az+\sqrt{a^2z^2-4}\right\}\right]}$$

$$\Rightarrow a\frac{z^2}{2} \mp a\int \sqrt{z^2-\left(\frac{2}{a}\right)^2}\,dz = 2u+b$$

$$\Rightarrow \frac{az^2}{2} \mp a\left[\frac{z}{2}\sqrt{z^2-\frac{4}{a^2}} - \frac{4}{a^2}\log\left(z+\sqrt{z^2-\frac{4}{a^2}}\right)\right] = 2(x+ay)+b$$

$$\Rightarrow \frac{az^2}{2} \mp a\left[\frac{z}{2a}\sqrt{a^2z^2-4} - \frac{4}{a^2}\log\left(z+\sqrt{z^2-\frac{4}{a^2}}\right)\right] = 2(x+ay)+b$$

This is the complete integral.